

hep-th/0103047  
 BRX-TH-484  
 HUTP-00/A052  
 BOW-PH-121

# Vacuum states of $\mathcal{N} = 1^*$ mass deformations of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ conformal gauge theories and their brane interpretations

Stephen G. Naculich<sup>1,a</sup>, Howard J. Schnitzer<sup>2,3,b,c</sup>, and Niclas Wyllard<sup>4,b</sup>

<sup>a</sup>*Department of Physics  
 Bowdoin College, Brunswick, ME 04011*

<sup>b,3</sup>*Martin Fisher School of Physics  
 Brandeis University, Waltham, MA 02454*

<sup>c</sup>*Lyman Laboratory of Physics  
 Harvard University, Cambridge, MA 02138*

## Abstract

We find the classical supersymmetric vacuum states of a class of  $\mathcal{N} = 1^*$  field theories obtained by mass deforming superconformal models with simple gauge groups and  $\mathcal{N} = 4$  or  $\mathcal{N} = 2$  supersymmetry. In particular, new classical vacuum states for mass-deformed  $\mathcal{N} = 4$  models with  $\text{Sp}(2N)$  and  $\text{SO}(N)$  gauge symmetry are found. We also derive the classical vacua for various mass-deformed  $\mathcal{N} = 2$  models with  $\text{Sp}(2N)$  and  $\text{SU}(N)$  gauge groups and antisymmetric (and symmetric) hypermultiplets. We suggest interpretations of the mass-deformed vacua in terms of three-branes expanded into five-brane configurations.

---

<sup>1</sup> Research supported in part by the NSF under grant no. PHY94-07194 through the ITP Scholars Program.

<sup>2</sup> Research supported in part by the DOE under grant DE-FG02-92ER40706.

<sup>3</sup> Permanent address.

<sup>4</sup> Research supported by the DOE under grant DE-FG02-92ER40706.  
 naculich@bowdoin.edu; schnitzer,wyllard@brandeis.edu

# 1 Introduction

Maldacena's conjecture relating string theories on anti-de Sitter (AdS) spaces to conformal field theories on their boundary [1] has provided a powerful laboratory to study supersymmetric theories, particularly at strong 't Hooft coupling. The best known example of the AdS/CFT correspondence is the duality between four-dimensional  $\mathcal{N} = 4$   $SU(N)$  Yang-Mills theory and type IIB string theory on  $AdS_5 \times S^5$ . To make contact with more realistic and less constrained four-dimensional field theories, it is of interest to find examples of gauge/gravity duals with reduced supersymmetry and broken superconformal invariance. Orbifold theories that correspond to projections of the  $\mathcal{N} = 4$   $SU(N)$  theory by a discrete subgroup  $\Gamma$  of the  $SU(4)$  R-symmetry group have been considered in refs. [2]. The resulting conformal field theories, which are dual to IIB string theory on  $AdS_5 \times S^5/\Gamma$ , have  $\mathcal{N} = 0, 1$ , or  $2$  supersymmetries, and the gauge group is generically of the form  $\prod_i U(N_i)$ . Another class of models with an AdS/CFT correspondence are orientifold models [3, 4, 5, 6, 7]. A detailed study of this correspondence for IIB configurations consisting of D3 branes in the background of an orientifold plane, and in some cases, a  $\mathbb{Z}_2$  orbifold and/or D7-branes, that give rise to four-dimensional  $\mathcal{N} = 2$  (or  $\mathcal{N} = 4$ ) gauge theories with at most two factors has been carried out in ref. [8], and the chiral primaries on the two sides of the correspondence matched.

Of particular interest for our present work are mass deformations of superconformal theories, that break the supersymmetry to  $\mathcal{N} = 1$ . (These are referred to as  $\mathcal{N} = 1^*$  theories.) Such systems naively lead to naked singularities on the gravity side [9]. However, a careful analysis by Polchinski and Strassler [10] revealed that the naked singularity is actually replaced by an expanded brane source. Other works on  $\mathcal{N} = 1$  gauge/gravity duals include [11].

In this paper, we will focus on  $\mathcal{N} = 1^*$  mass-deformations of four-dimensional superconformal gauge theories with simple gauge groups and  $\mathcal{N} = 4$  or  $\mathcal{N} = 2$  supersymmetry. The undeformed conformal theories arise as the low-energy effective theories on a stack of D3-branes in various orientifold and orbifold backgrounds in type IIB string theory (see ref. [8] for a detailed description). We will identify the classical vacuum states of these theories, and propose corresponding D-brane interpretations.

In section 2, we review the classical vacuum states of the mass-deformed  $\mathcal{N} = 4$   $SU(N)$  Yang-Mills theory, *i.e.*, the  $\mathcal{N} = 1^*$  theory studied in ref. [10]. The massive classical vacuum states of the mass-deformed  $\mathcal{N} = 4$  Yang-Mills theories with  $Sp(2N)$  and  $SO(N)$  gauge groups were initially analyzed and interpreted by Aharony and Rajaraman [12]. In section 3, we present new classical vacuum states (which preserve a subset of the gauge symmetry) for these theories, and propose brane configurations corresponding to these new solutions. Section 4 is devoted to the  $\mathcal{N} = 1^*$  mass-deformation of the  $\mathcal{N} = 2$   $Sp(2N)$  gauge theory with one hypermultiplet in the antisymmetric representation and four hypermultiplets in the fundamental representation, which corresponds to D3-branes in the presence of an O7-plane and D7-branes. The vacuum states of this mass-deformed theory are given a brane interpretation. In section 5 we study the mass-deformation of models corresponding to D3-branes in a  $\mathbb{Z}_2$ -orbifold background and an orientifold. The two resulting  $\mathcal{N} = 2$  theories with simple gauge group are (i)  $SU(N)$  with a hypermultiplet in the antisymmetric and symmetric representation, and (ii)  $SU(N)$  with two hypermultiplets in the antisymmetric representation and four in the fundamental representation. The classical vacuum states of

these  $\mathcal{N} = 1^*$  theories are found, and brane interpretations proposed. Our conventions and some technical details are collected in two appendices.

## 2 Review of the mass-deformed $\mathcal{N} = 4$ SU( $N$ ) model

In this section, we review the mass-deformed  $\mathcal{N} = 4$  SU( $N$ ) gauge theory known as the  $\mathcal{N} = 1^*$  theory [13], and the interpretation of its vacua in the brane picture [14] and dual supergravity theory [10]. The superpotential of the model, written in terms of  $\mathcal{N} = 1$  superfields, is

$$\mathcal{W} = \frac{2\sqrt{2}}{6g_{\text{YM}}^2} \epsilon^{ijk} \text{tr}(\phi^i [\phi^j, \phi^k]), \quad (2.1)$$

where  $\phi^i$  ( $i = 1, 2, 3$ ) are chiral superfields transforming in the adjoint representation of SU( $N$ ). To this superpotential, we add the mass deformation

$$\mathcal{W}_{\text{mass}} = \frac{\sqrt{2}}{g_{\text{YM}}^2} \sum_{i=1}^3 m_i \text{tr}(\phi^i)^2, \quad (2.2)$$

which, when all the masses are non-zero, breaks the supersymmetry down to  $\mathcal{N} = 1$ . In this case, one can rescale the  $\phi^i$ 's to make the masses equal. In what follows, we concentrate on this case, denoting the common (real) mass by  $m$ . The classical supersymmetric vacuum states are obtained by solving the F- and D-term equations, which are

$$[\phi^i, \phi^j] = -m\epsilon^{ijk}\phi^k, \quad (2.3)$$

and

$$\sum_{i=1}^3 [\phi^i, (\phi^i)^\dagger] = 0, \quad (2.4)$$

respectively. Equation (2.3) together with (2.4) imply that the  $\phi^i$ 's are anti-hermitian [15]. The most general solution is

$$\phi^i = mT^i, \quad (2.5)$$

where the  $T^i$ 's form an  $N$ -dimensional (in general reducible) representation of the su(2) Lie algebra (A.1). It is always possible to choose a block-diagonal basis,

$$T^i = \begin{pmatrix} T_{n_1}^i & & \\ & \ddots & \\ & & T_{n_l}^i \end{pmatrix}, \quad \sum_{k=1}^l n_k = N, \quad (2.6)$$

in which  $T_{n_k}^i$  are the generators of the  $n_k$ -dimensional irreducible representation of su(2) in standard form (A.2), (A.3), with  $T^1 = T^x$ ,  $T^2 = T^y$ , and  $T^3 = T^z$ .

The undeformed  $\mathcal{N} = 4$  SU( $N$ ) gauge theory can be realized on a stack of  $N$  D3-branes in a flat background. We take the D3-branes to span the 0123 directions. The positions of the D-branes in the transverse directions  $z^i$  (where we define  $z^3 = x^4 + ix^5$ ,  $z^1 = x^6 + ix^7$ , and  $z^2 = x^8 + ix^9$ ) are represented by the complex scalar fields  $\phi^i$ , which are mutually commuting in the vacuum state of the undeformed theory.

In the mass-deformed theory, the D3-branes are polarized by the Myers effect [14] (see also [16]). Since the generators  $T_{n_k}^i$  satisfy the Casimir relation

$$(T_{n_k}^1)^2 + (T_{n_k}^2)^2 + (T_{n_k}^3)^2 = c_2(n_k)\mathbb{1}_{n_k}, \quad (2.7)$$

where  $c_2(n_k) = -\frac{1}{4}(n_k^2 - 1)$ , each block in the vacuum solution (2.5), (2.6) can be interpreted as the equation for a non-commutative two-sphere (in the 579 directions of the  $\mathbb{R}^6$  transverse to the D3-branes). In the dual  $AdS_5 \times S^5$  supergravity theory, it was shown [10] that the vacuum solution (2.6) corresponds to the addition of a set of D5-branes to the  $AdS_5 \times S^5$  background, each with topology  $\mathbb{R}^4 \times S^2$ , where the radius of the  $k$ th D5-brane is proportional to  $n_k$ , the dimension of the  $k$ th block in the field theory solution. (The supergravity approximation is valid provided all the  $n_k$  are large.)

Support for this interpretation is obtained by comparing the gauge enhancement on the two sides of the correspondence. On the field theory side, the unbroken gauge symmetry for the solution (2.6) is generated by all  $SU(N)$  matrices  $U$  satisfying  $U^\dagger \phi^i U = \phi^i$ . When there are  $m$  irreducible representations of the same dimension, one gets  $U(m)$  gauge enhancement; the total gauge enhancement is the product of all such factors divided by an overall  $U(1)$ . In particular, when all the blocks are of different dimensions, the gauge symmetry is broken to an abelian subgroup. This is in agreement with the supergravity side, in which each stack of  $m$  coincident five-branes contributes an  $U(m)$  factor; the total gauge enhancement is the product of all such factors divided by an overall  $U(1)$ . In ref. [10], the above heuristic arguments were firmly established by solving the supergravity field equations to first order, in an expansion based on the smallness of the D5-brane charge relative to the D3-brane charge.

### 3 $\mathcal{N} = 4$ orientifold models: $Sp(2N)$ and $SO(N)$

There are two other classes of  $\mathcal{N} = 4$  superconformal gauge theories with simple gauge groups, namely  $Sp(2N)$  and  $SO(N)$ . These theories are realized on a stack of D3 branes in the background of an O3 orientifold plane. In this section we consider the  $\mathcal{N} = 1^*$  mass deformation of these models, first within the context of the supersymmetric field theory, and then from the brane perspective. These  $\mathcal{N} = 1^*$  theories were studied in the context of the AdS/CFT correspondence by Aharony and Rajaraman [12], who discussed the supergravity interpretation of some [15] (see also [17]), but not all, of the gauge theory vacuum states. We will exhibit a new class of gauge theory vacua, and then discuss their brane interpretations.

#### 3.1 Field theory vacuum solutions

##### 3.1.1 $\mathcal{N} = 4$ $Sp(2N)$

The superpotential and mass deformation of the  $\mathcal{N} = 4$   $Sp(2N)$  gauge theory are the same as in the  $SU(N)$  theory (2.1), (2.2), yielding identical F- and D-term equations (2.3), (2.4). In addition, however, the  $\phi^i$  must be  $2N \times 2N$  matrices that satisfy

$$J(\phi^i)^T J = \phi^i, \quad (3.1)$$

which defines the generators of the adjoint representation of  $\text{Sp}(2N)$ . The real matrix  $J$  is the symplectic unit of  $\text{Sp}(2N)$ , satisfying  $J^T = -J$  and  $J^2 = -\mathbb{1}_{2N}$ ; the standard basis is

$$J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}. \quad (3.2)$$

The general solution to the F- and D-term equations is, as before,

$$\phi^i = mT^i, \quad (3.3)$$

where the  $T^i$ 's generate a  $2N$ -dimensional (in general reducible) representation of  $\text{su}(2)$ . By a unitary transformation,  $\phi^i \rightarrow U\phi^i U^\dagger$ , the  $T^i$ 's may be brought to block-diagonal form (2.6). In this basis, the  $\text{Sp}(2N)$  condition (3.1) becomes

$$g^*(\phi^i)^T g = \phi^i \quad \Rightarrow \quad (\phi^i)^T g = -g\phi^i, \quad (3.4)$$

where  $g = U^T J U$ , and hence  $g^T = -g$  and  $gg^* = -\mathbb{1}_{2N}$ . While the  $\phi^i$  are block-diagonal matrices in this basis, the matrix  $g$  need not be.

The classically *massive* vacua, *i.e.*, those in which the gauge symmetry is completely broken, were completely classified in ref. [15], and their dual supergravity interpretation found in ref. [12]. One classically massive vacuum was found for every partition of  $2N$  into *distinct, even* integers. In other words, the massive vacua correspond to solutions (3.3), (2.6) in which all the irreducible blocks have different, even dimensions. One may understand this result as follows: first, breaking of the symmetry to an abelian subgroup requires the irreducible blocks in  $T^i$  to be distinct. In this case, it may be shown (see appendix B) that the matrix  $g$  must be block-diagonal. With the irreducible representations written in the standard basis (A.2), (A.3), it may be further shown that (with the convention that  $a_k = a_{-k}$  in (A.3)) within each block,  $g$  is proportional to

$$K = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \ddots & & & \end{pmatrix}, \quad (3.5)$$

with signs alternating down the reverse-diagonal, and all other entries vanishing. The matrix  $K$  (and therefore  $g$ ) is antisymmetric only when the dimension of the representation is even; hence, all the irreducible blocks in (2.6) must be even-dimensional. These solutions break the gauge group completely, as we will see below.

As noted in ref. [12], the above solutions – direct sums of distinct, even-dimensional representations of  $\text{su}(2)$  – do not exhaust the classical vacua of the mass-deformed  $\mathcal{N} = 4$   $\text{Sp}(2N)$  theory; vacua in which the  $\text{Sp}(2N)$  gauge symmetry is only partially broken are not included in this class. We now present another class of classical vacua of the  $\text{Sp}(2N)$  gauge theory.

Consider a solution (3.3) in which two of the irreducible blocks have the same (not necessarily even) dimension  $n$ ,

$$T^i = \begin{pmatrix} T_n^i & 0 \\ 0 & T_n^i \end{pmatrix}. \quad (3.6)$$

In this case, the part of the matrix  $g$  acting on these two blocks need not be block-diagonal, but has the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ -g_{12}^T & g_{22} \end{pmatrix}, \quad (3.7)$$

where the  $n \times n$ -dimensional matrices  $g_{ij}$  must each be proportional to  $K$ , as shown in appendix B. If  $n$  is odd, then  $g_{11} = g_{22} = 0$ , and by a phase transformation,  $g$  may put into the form

$$g = \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix}. \quad (3.8)$$

If  $n$  is even, it is still possible to bring  $g$  into the form (3.8) by a transformation that does not affect  $T^i$ . Finally,  $g = U^T J U$  is solved by  $U = \text{diag}(\mathbb{1}_n, K)$ . This allows the unitary transformation to be undone, and using the fact that  $K T_n^i K^T = -(T_n^i)^T$ , we obtain

$$\phi^i = m \begin{pmatrix} T_n^i & 0 \\ 0 & (-T_n^i)^T \end{pmatrix}, \quad (3.9)$$

which satisfies (3.1), as may be directly verified.

Thus, in addition to the massive vacua discussed in ref. [12], the mass-deformed  $\mathcal{N} = 4$   $\text{Sp}(2N)$  gauge theory has vacua corresponding to any pair of identical irreducible (not necessarily even-dimensional)  $\text{su}(2)$  representations. If the representations are even-dimensional, then by a change of basis it is possible to diagonalize  $g$ , *i.e.*, put it in the form  $\text{diag}(K, K)$ , but for odd-dimensional representations,  $g$  must act to exchange the pair.

There also exist vacua corresponding to solutions containing the direct sum of an arbitrary number  $m$  of identical even-dimensional irreducible representations, in which case  $g$  can be put in the form  $\text{diag}(K, K, \dots, K)$ , and solutions comprising  $2m$  identical odd-dimensional irreducible representations, in which case  $g$  can be written as the  $m \times m$  unit matrix tensored with (3.8). The most general vacuum solution consists of a direct sum of these two types of reducible representations, of varying dimensions (see appendix B for more details).

#### *Unbroken symmetry group of the vacuum solutions*

Let us now discuss the gauge enhancements for each of the solutions above, *i.e.*, the subgroup of the original  $\text{Sp}(2N)$  that remains unbroken by the scalar field vacuum expectation values  $\phi^i = m T^i$ . We seek all matrices  $U \in \text{Sp}(2N)$  (which obey  $U^\dagger U = \mathbb{1}$  and  $U^T J U = J$ ) such that  $U^\dagger T^i U = T^i$ . Infinitesimally we obtain the conditions (writing  $U = e^H$ ):  $H^\dagger = -H$ ,  $H^T J + J H = 0$ , and  $[H, T^i] = 0$ .

First, if  $T^i$  is an *irreducible* representation, then  $[H, T^i] = 0$  implies  $H \propto \mathbb{1}$ , by Schur's lemma. Combining this with  $H^T J + J H = 0$  shows that  $H = 0$ ; thus, the gauge group is completely broken.

Next, let  $T^i$  be reducible into *two* irreducible representations

$$\begin{pmatrix} T_{n_1}^i & 0 \\ 0 & T_{n_2}^i \end{pmatrix}. \quad (3.10)$$

In the block-diagonal basis (3.10), the condition  $J H^T J = H$  becomes

$$g^* H^T g = H. \quad (3.11)$$

Write  $H$  as  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . If the irreducible representations are distinct (and therefore, as shown above, even-dimensional), then  $[H, T^i] = 0$  implies  $b = c = 0$  by Schur's lemma, and  $a$  and  $d$  are proportional to the unit matrix. The matrix  $g$  is block-diagonal, with each block proportional to  $K$ . Hence (3.11) implies that  $a = d = 0$ , so again there is no unbroken gauge symmetry. Similarly, for the direct sum of an arbitrary number of *distinct* (even-dimensional) irreducible representations, the gauge group is completely broken; all these solutions therefore correspond to massive vacua, as asserted above.

If the two irreducible representations are identical (and have dimension  $n$ ), then Schur's lemma implies that  $a, b, c$ , and  $d$  are all proportional to the unit matrix, that is,  $H = h \otimes \mathbb{1}_n$ , where  $h$  is a  $2 \times 2$  matrix. The condition  $H^\dagger = -H$  implies that  $h$  is anti-hermitian.

If the two identical blocks are *odd-dimensional*, then  $K^T = K$  and  $K^2 = \mathbb{1}_n$ , so that  $g = i\sigma_y \otimes K$ , using (3.5) and (3.8). Equation (3.11) then implies that  $(i\sigma_y)h^T(i\sigma_y) = h$ , that is,  $h \in \mathfrak{sp}(2)$ . Thus, two equal odd-dimensional blocks give rise to  $\mathrm{Sp}(2)$  gauge enhancement. Generalizing to  $2m$  identical odd-dimensional blocks, one finds that  $H = h \otimes \mathbb{1}_n$ , where  $h$  is now a  $2m \times 2m$ -dimensional (anti-hermitian) matrix. In this basis,  $g$  can be written as  $j \otimes K$  where  $j$  is of the form  $\mathbb{1}_m \otimes i\sigma_y$ . Then (3.11) yields  $jh^Tj = h$ , that is,  $h \in \mathfrak{sp}(2m)$ . Hence, when  $T^i$  consists of  $2m$  identical blocks of odd dimension, the unbroken gauge symmetry is  $\mathrm{Sp}(2m)$ .

If the two identical blocks are *even-dimensional*, then, by a change of basis that does not affect the form (3.6),  $g$  may be written  $\mathbb{1}_2 \otimes K$ , where now  $K^T = -K$  and  $K^2 = -\mathbb{1}_n$ . Equation (3.11) then implies that  $h^T = -h$ , *i.e.*,  $h \in \mathfrak{so}(2)$ . Thus, with two identical even-dimensional blocks, the gauge enhancement is  $\mathrm{SO}(2)$ . The argument easily generalizes to  $m$  identical even-dimensional blocks, yielding  $\mathrm{SO}(m)$  gauge enhancement, as noted in ref. [12].

In summary, for  $2m$  odd-dimensional (resp.  $m$  even-dimensional) blocks of the same dimension, the gauge enhancement is  $\mathrm{Sp}(2m)$  (resp.  $\mathrm{SO}(m)$ ).<sup>5</sup>

### 3.1.2 $\mathcal{N} = 4$ $\mathrm{SO}(N)$

The superpotential and mass deformation of the  $\mathcal{N} = 4$   $\mathrm{SO}(N)$  gauge theory are the same as in the  $\mathrm{SU}(N)$  theory (2.1), (2.2), yielding identical F- and D-term equations (2.3), (2.4). In addition, however, the  $N \times N$  matrices  $\phi^i$  must satisfy

$$(\phi^i)^T = -\phi^i, \quad (3.12)$$

appropriate to the generators of the adjoint representation of  $\mathrm{SO}(N)$ .

The general solution to the F- and D-term equations is, as before,

$$\phi^i = mT^i, \quad (3.13)$$

where the  $T^i$ 's generate a  $N$ -dimensional (in general reducible) representation of  $\mathfrak{su}(2)$ . By a unitary transformation,  $\phi^i \rightarrow U\phi^iU^\dagger$ , the  $T^i$ 's may be brought to block-diagonal form (2.6). In this basis, the  $\mathrm{SO}(N)$  condition (3.12) becomes

$$f^*(\phi^i)^T f = -\phi^i \quad \Rightarrow \quad (\phi^i)^T f = -f\phi^i, \quad (3.14)$$

---

<sup>5</sup>In this paper, we are only discussing the classical vacua. Quantum effects may cause a further splitting of the vacua and consequent reduction of the gauge symmetry.

where  $f = U^T U$ , and hence  $f^T = f$  and  $ff^* = \mathbb{1}_N$ . While the  $\phi^i$  are block-diagonal matrices in this basis, the matrix  $f$  need not be.

The classically *massive* vacua, *i.e.*, those in which the gauge symmetry is completely broken, were completely classified in ref. [15], and their dual supergravity interpretation found in ref. [12]. One classically massive vacuum was found for every partition of  $N$  into *distinct* odd integers. In other words, the massive vacua correspond to solutions (3.13), (2.6) in which all the irreducible blocks have different, odd dimensions. One may understand this result as follows: first, breaking of the symmetry to an abelian subgroup requires the irreducible blocks in  $T^i$  to be distinct. As in the previous section, it may be shown that the matrix  $f$  must be block-diagonal, and within each block,  $f$  is proportional to  $K$  (3.5).  $K$  is symmetric only when the dimension of the representation is odd; hence, all the irreducible blocks in (2.6) must be odd-dimensional. When the blocks are odd-dimensional, it is possible to find a block-diagonal unitary matrix  $U$  that rotates the  $T^i$ 's in each block into real antisymmetric matrices (*i.e.*,  $\text{so}(N)$  matrices). These solutions break the gauge group completely, as we will see below.

As noted in [12], the above solutions – direct sums of distinct, odd-dimensional representations of  $\text{su}(2)$  – do not exhaust the classical vacua of the mass-deformed  $\mathcal{N} = 4$   $\text{SO}(N)$  theory; vacua in which the gauge symmetry is only partially broken are not included in this class.<sup>6</sup> We now present another class of classical vacua of the  $\text{SO}(N)$  gauge theory.

Consider a solution of the form (3.6) in which two of the irreducible blocks have the same (not necessarily odd) dimension  $n$ . The part of the matrix  $f$  acting on these two blocks need not be block-diagonal, but has the form

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{12}^T & f_{22} \end{pmatrix}, \quad (3.15)$$

where the  $n \times n$ -dimensional matrices  $f_{ij}$  must each be proportional to  $K$  (3.5). When  $n$  is even,  $f_{11} = f_{22} = 0$ , and by a phase transformation,  $f$  may be put into the form

$$f = \begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix}. \quad (3.16)$$

When  $n$  is odd,  $f$  may still be brought into the form (3.16) by a unitary transformation that does not affect  $T^i$ . Then,  $f = U^T U$  is solved by

$$U = \frac{1}{\sqrt{2i}} \begin{pmatrix} \mathbb{1} & iK \\ iK^T & \mathbb{1} \end{pmatrix}, \quad (3.17)$$

allowing the unitary transformation to be undone, yielding

$$\phi^i = \frac{m}{2} \begin{pmatrix} T_n^i - T_n^{iT} & -i(T_n^i + T_n^{iT})K \\ iK^T(T_n^i + T_n^{iT}) & T_n^i - T_n^{iT} \end{pmatrix}. \quad (3.18)$$

This is manifestly antisymmetric, in accord with (3.12). Specifically, we have

$$\phi^{1,3} = m \begin{pmatrix} 0 & -iT_n^{x,z}K \\ iK^T T_n^{x,z} & 0 \end{pmatrix}, \quad \phi^2 = m \begin{pmatrix} T_n^y & 0 \\ 0 & T_n^y \end{pmatrix}. \quad (3.19)$$

---

<sup>6</sup> One example cited in [12] is  $\text{SO}(6) \rightarrow \text{SU}(2) \times \text{U}(1)$ .



Using (A.2) and (A.3), one sees that in this basis the  $\phi^i$  are all real and antisymmetric.

Thus, in addition to the massive vacua discussed in ref. [12], the mass-deformed  $\mathcal{N} = 4$   $SO(N)$  gauge theory has vacua corresponding to any pair of identical irreducible (not necessarily odd-dimensional)  $su(2)$  representations. If the representations are odd-dimensional, then by a change of basis it is possible to diagonalize  $f$ , *i.e.*, put it in the form  $\text{diag}(K, K)$ , but for even-dimensional representations,  $f$  must act to exchange the pair.

There also exist vacua corresponding to solutions containing the direct sum of an arbitrary number  $m$  of identical odd-dimensional irreducible representations, in which case  $f$  can be unitarily transformed into the form  $\text{diag}(K, K, \dots, K)$ , and solutions comprising  $2m$  identical even-dimensional irreducible representations, in which case  $f$  can be written as the  $m \times m$  unit matrix tensored with (3.16). The most general vacuum solution consists of a direct sum of these two types of reducible representations, of varying dimensions.

#### *Unbroken symmetry group of the vacuum solutions*

Let us now discuss the gauge enhancements for the solutions above, *i.e.*, the subgroup of the original  $SO(N)$  that remains unbroken by the scalar field vacuum expectation values  $\phi^i = mT^i$ . The discussion is similar to that for  $Sp(2N)$ . We seek all matrices  $U \in SO(N)$  (which obey  $U^T U = \mathbb{1}$  and  $U^* = U$ ) that satisfy  $U^T T^i U = T^i$ . Writing  $U = e^H$ , these conditions become  $H^T = -H$ ,  $H^* = H$ , and  $[H, T^i] = 0$ .

As before, if  $T^i$  is irreducible, or reducible into *distinct* (odd-dimensional) representations, the symmetry group is completely broken.

If  $T^i$  is reducible into two identical representations of dimension  $n$ , then  $H = h \otimes \mathbb{1}_n$ . In the block-diagonal basis (3.6), the condition  $H^T = -H$  becomes

$$f^* H^T f = -H, \quad (3.20)$$

and  $H^* = H$  transforms into a condition that, together with eq. (3.20), implies that  $h$  is anti-hermitian.

If the two identical blocks are *even-dimensional*, then  $K^T = -K$  and  $K^2 = -\mathbb{1}_n$ , so that  $f = i\sigma_y \otimes K$ , by (3.5) and (3.16). Equation (3.20) then implies that  $(i\sigma_y)h^T(i\sigma_y) = h$ , that is,  $h \in \mathfrak{sp}(2)$ , giving  $Sp(2)$  gauge enhancement. Generalizing to  $2m$  identical even-dimensional blocks, following the reasoning in section 3.1.1, we find that the unbroken symmetry is  $Sp(2m)$ .

If the two identical blocks are *odd-dimensional*, then, by a change of basis that does not affect the form (3.6),  $f$  may be written  $\mathbb{1}_2 \otimes K$ , where now  $K^T = K$  and  $K^2 = \mathbb{1}_n$ . Equation (3.20) then implies that  $h^T = -h$ , yielding  $SO(2)$  gauge enhancement. Generalizing to  $m$  identical odd-dimensional blocks, we find that the unbroken symmetry is  $SO(m)$ , as noted in ref. [12].

In summary, for  $2m$  even-dimensional (resp.  $m$  odd-dimensional) blocks of the same dimension, the gauge enhancement is  $Sp(2m)$  (resp.  $SO(m)$ ).<sup>5,7</sup>

---

<sup>7</sup> The symmetry-breaking pattern mentioned in the previous footnote corresponds to taking two 2-dimensional and two 1-dimensional  $su(2)$  representations, which yields the unbroken gauge group  $Sp(2) \times SO(2) \cong SU(2) \times U(1)$ .

### 3.2 Brane interpretation of the vacua

The  $\mathcal{N} = 4$   $\text{Sp}(2N)$  and  $\text{SO}(N)$  gauge theories are realized as the low-energy effective theories on a stack of D3-branes probing a type IIB background with orientifold group generated by  $\Omega' = \Omega(-1)^{F_L} R_{456789}$ , where  $\Omega$  is the world-sheet parity operation,  $(-1)^{F_L}$  reverses the sign of the left-moving Ramond sector, and  $R_{456789}$  changes the sign of the coordinates transverse to the D3-branes. This corresponds to the presence of an orientifold 3-plane at the origin of the coordinates transverse to the D3-branes.

The bosonic fields of these models may be obtained as projections of the gauge fields  $A^\mu$  and adjoint scalars  $\phi^i$  of the  $\mathcal{N} = 4$   $\text{SU}(N)$  model as follows. Consider a stack of  $N$  D3-branes in an orientifold background, where the orientifold group acts on the D3-branes via the  $N \times N$  matrix  $\gamma_{\Omega'}$  [18], so that the fields are projected via

$$A^\mu = -\gamma_{\Omega'}(A^\mu)^T \gamma_{\Omega'}^{-1}, \quad \phi^i = -\gamma_{\Omega'}(\phi^i)^T \gamma_{\Omega'}^{-1}. \quad (3.21)$$

The two different gauge groups arise from different choices of how the NS-NS and R-R 3-forms are twisted under the orientifolding [5], and correspond to choices of discrete torsion. For  $\text{Sp}(N)$  (where  $N$  must be even),  $\gamma_{\Omega'} = J$ , leading to the restrictions (3.1) on the adjoint scalars. For  $\text{SO}(N)$ ,  $\gamma_{\Omega'} = \mathbb{1}$ , leading to the restrictions (3.12) on the adjoint scalars.

First, consider the interpretation of the vacuum solutions of the mass-deformed theory consisting of *distinct* irreducible representations. On the covering space of the orientifold, each block of the solution corresponds to a two-sphere, as in the  $\text{SU}(N)$  theory. For these solutions, the projection matrix  $g$  (or  $f$ ) is block-diagonal, so the orientifold acts within each block, geometrically projecting each  $S^2$  to  $\mathbb{RP}^2$ .

Aharony and Rajaraman analyzed the dual supergravity theory [12], with the near-horizon geometry  $AdS_5 \times \mathbb{RP}^5$ , in which these mass-deformed vacuum solutions correspond to 5-branes of topology  $\mathbb{R}^4 \times \mathbb{RP}^2$ , where the  $\mathbb{RP}^2$  is embedded inside the  $\mathbb{RP}^5$  factor, and the  $\mathbb{R}^4$  lies inside  $AdS_5$ . Supporting evidence for this interpretation was obtained via an analysis of the D3-brane charges carried by the D5-branes, and the fluxes of the gauge field and the NS-NS 2-form.

What is the brane interpretation of the other solutions (3.6) found above? On the double cover, the pair of identical blocks corresponds to a pair of two-spheres with equal radius.

When the representations are odd-dimensional (even-dimensional) in the  $\text{Sp}$  ( $\text{SO}$ ) gauge theory, the action of the orientifold group necessarily exchanges the two blocks (see eqs. (3.8) and (3.16)), so the pair of spheres are actually mirror images. We conclude that our new solutions are to be interpreted as a single physical D5-brane with topology  $\mathbb{R}^4 \times S^2$ , *i.e.*, one D5-brane and its mirror. In retrospect, the presence of such a solution is not surprising; the configuration it describes arises from a five-brane mirror pair away from the orientifold plane being brought to the orientifold plane. More generally, a solution containing  $2m$  identical odd-dimensional (even-dimensional) representations corresponds to  $m$  D5-branes, whose associated  $S^2$ 's have equal radii, and their mirrors. As we saw above, the gauge enhancement for such a solution is  $\text{Sp}(2m)$ . This is analogous to the  $\text{Sp}(2m)$  gauge enhancement of  $2m$  coincident D3-branes, of which the first  $m$  are mapped to the second  $m$  under the orientifold, that is,  $\gamma_{\Omega'} = J$ .

When the representations are even-dimensional (odd-dimensional) in the  $\text{Sp}$  ( $\text{SO}$ ) gauge theory, the orientifold action in (3.8) and (3.16) still exchanges the two blocks, so these solu-

tions may also be thought of in terms of a brane-mirror pair of  $S^2$ 's. However, in this case  $g$  and  $f$  can also be transformed to the block-diagonal form  $\text{diag}(K, K)$ . In this basis, each  $S^2$  is mapped to itself by the orientifold symmetry, and we end up with a pair of  $\mathbb{RP}^2$ 's. More generally, a solution containing  $m$  identical even-dimensional (odd-dimensional) representations corresponds to  $m$   $\mathbb{RP}^2$ 's of equal radius. As we saw above, the gauge enhancement for this solution is  $\text{SO}(m)$ . This is analogous to the  $\text{SO}(m)$  gauge enhancement of  $m$  coincident fractional D3-branes, when each is mapped to itself under the orientifold symmetry ( $\gamma_{\Omega'} = \mathbb{1}$ ). The two different descriptions discussed above are analogous to the usual transformation of two fractional branes into a single physical brane, *i.e.*, a brane-mirror pair.

## 4 $\mathcal{N} = 2$ orientifold model: $\text{Sp}(2N) + \square + 4 \square$

In this section, we turn to  $\mathcal{N} = 2$  superconformal gauge theories with a simple gauge group. There is only one such theory that arises from IIB string theory in an orientifold (only) background, namely,  $\mathcal{N} = 2$   $\text{Sp}(2N)$  gauge theory with one antisymmetric and four fundamental hypermultiplets [19]. The field theory contains (in  $\mathcal{N} = 1$  language) one vector multiplet, one hypermultiplet  $\phi_a^b$  in the adjoint representation of  $\text{Sp}(2N)$ , one hypermultiplet  $A_{ab}$  in the antisymmetric representation, and one hypermultiplet  $\tilde{A}^{ab}$  in the conjugate antisymmetric representation. In addition there are four hypermultiplets  $Q_a^I$  in the fundamental representation, and four hypermultiplets  $\tilde{Q}^{Ia}$  in the conjugate fundamental representation. Indices may be raised and lowered using the symplectic unit  $J^{ab}$  and its inverse  $J_{ab}$ . Any representation is equivalent to its conjugate via the raising or lowering of indices using  $J$ ; for example,  $\tilde{A}_{ab} = J_{ac}J_{bd}\tilde{A}^{cd}$  transforms in the antisymmetric representation. As a result of (3.1), the adjoint representation written with both indices down, *i.e.*,  $\phi_{ab} = J_{bc}\phi_a^c$ , is symmetric.

The  $\mathcal{N} = 2$  superpotential of this theory is

$$\mathcal{W} = \frac{2\sqrt{2}}{g_{\text{YM}}^2} [-2\tilde{A}^{ab}\phi_b^c A_{ca} + \tilde{Q}^{aI}\phi_a^b Q_b^I], \quad \begin{array}{ll} a, b, c &= 1 \text{ to } 2N, \\ I &= 1 \text{ to } 4, \end{array} \quad (4.1)$$

to which we add the mass deformation

$$\mathcal{W}_{\text{mass}} = \frac{\sqrt{2}}{g_{\text{YM}}^2} \left\{ m[\phi_a^b \phi_b^a + A^{ab} A_{ab} + \tilde{A}^{ab} \tilde{A}_{ab}] + M Q_a^I \tilde{Q}^{Ia} \right\}. \quad (4.2)$$

Finding the vacuum solutions will be facilitated by defining

$$\begin{aligned} (\phi^1)_a^b &= J^{bc} A_{ac}, & s_a^b &= Q_a^I \tilde{Q}^{Ib}, \\ (\phi^2)_a^b &= J_{ac} \tilde{A}^{cb}, & t_a^b &= \tilde{Q}_a^I Q^{Ib}, \\ (\phi^3)_a^b &= \phi_a^b, \end{aligned} \quad (4.3)$$

where, of course,  $s$  and  $t$  are not independent. By virtue of the symmetry of  $\phi_{ab}$ , the antisymmetry of  $A_{ab}$  and  $\tilde{A}^{ab}$ , and the definitions of  $s_a^b$  and  $t_a^b$ , these fields are constrained to satisfy

$$\begin{aligned} J(\phi^1)^T J &= -\phi^1, & Js^T J &= t, \\ J(\phi^2)^T J &= -\phi^2, & Jt^T J &= s, \\ J(\phi^3)^T J &= \phi^3. \end{aligned} \quad (4.4)$$

The superpotential (4.1) and mass deformation (4.2) may be rewritten in terms of the fields (4.3) as

$$\begin{aligned}\mathcal{W} &= \frac{2\sqrt{2}}{g_{\text{YM}}^2} \text{tr}(\phi^1[\phi^2, \phi^3] + s\phi^3), \\ \mathcal{W}_{\text{mass}} &= \frac{\sqrt{2}}{g_{\text{YM}}^2} \text{tr}(m\phi^i\phi^i + Ms),\end{aligned}\tag{4.5}$$

giving rise to the F-term equations

$$\begin{aligned}[\phi^1, \phi^2] &= -m\phi^3 - \frac{1}{2}(s+t), \\ [\phi^2, \phi^3] &= -m\phi^1, \\ [\phi^3, \phi^1] &= -m\phi^2, \\ [\phi^3, s] &= [\phi^3, t] = 0, \\ \phi^3 s &= -\frac{1}{2}Ms, \\ \phi^3 t &= \frac{1}{2}Mt.\end{aligned}\tag{4.6}$$

We now turn to the classical vacua of this theory by solving the F-terms equations (4.6) subject to the restrictions (4.4).

## 4.1 Field theory vacuum solutions

### 4.1.1 Case 1: $Q = \tilde{Q} = 0$

Consider the branch of moduli space in which the expectation values of the scalars in the fundamental hypermultiplets  $Q_a^I$  and  $\tilde{Q}^{Ia}$  vanish (thus  $s = t = 0$ ). The F-term equations (4.6) then reduce to those encountered in previous sections. The general solution, which also satisfies the D-term equations, can be written as

$$\phi^i = mT^i = m \begin{pmatrix} T_{n_1}^i & & \\ & \ddots & \\ & & T_{n_l}^i \end{pmatrix},\tag{4.7}$$

where, as before, the  $T_{n_k}^i$ 's are generators of the irreducible representations of  $\mathfrak{su}(2)$ . In this block-diagonal basis, the constraints (4.4) become

$$g^*(\phi^1)^T g = -\phi^1, \quad g^*(\phi^2)^T g = -\phi^2, \quad g^*(\phi^3)^T g = \phi^3,\tag{4.8}$$

where  $g = U^T J U$ , which is not necessarily block-diagonal. We must determine *which* of the solutions (4.7) satisfy the constraints (4.8), which differ from the constraints (3.4) of the  $\mathcal{N} = 4$  Sp(2N) theory.

We first show that this theory has no *classically massive* vacua, *i.e.*, vacua in which the gauge symmetry is completely broken. Such vacua would correspond to solutions (4.7) in which all the irreducible blocks had different dimensions. In this case, it may be shown (see

appendix B) using (4.8) that  $g$  must be block-diagonal, and within each block, proportional to

$$K' = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ \ddots & & & \end{pmatrix}. \quad (4.9)$$

This matrix is symmetric, however, and so cannot be written as  $g = U^T J U$ , which obeys  $g^T = -g$ . Thus, no solution consisting of a direct sum of *distinct* irreducible representations is possible. In particular, a single irreducible representation is not possible.

Consider a solution (3.6) in which two of the irreducible blocks have the same dimension  $n$ . It may be shown, using (4.8), that the part of the matrix  $g$  acting on these two blocks has the form

$$g = \begin{pmatrix} 0 & K' \\ -K' & 0 \end{pmatrix}. \quad (4.10)$$

Then  $g = U^T J U$  is solved by  $U = \text{diag}(1, K')$ , allowing the unitary transformation to be undone. Using the fact that  $K' T^z K' = (-T^z)^T$  and  $K' T^{x,y} K' = (+T^{x,y})^T$ , we obtain

$$\phi^1 = m \begin{pmatrix} T_n^x & 0 \\ 0 & (T_n^x)^T \end{pmatrix}, \quad \phi^2 = m \begin{pmatrix} T_n^y & 0 \\ 0 & (T_n^y)^T \end{pmatrix}, \quad \phi^3 = m \begin{pmatrix} T_n^z & 0 \\ 0 & (-T_n^z)^T \end{pmatrix}, \quad (4.11)$$

which satisfies the constraints (4.4), as may be directly verified.

Thus, while a single irreducible representation of  $\mathfrak{su}(2)$ , or a direct sum of distinct irreducible representations, do not correspond to classical vacua of the  $\mathcal{N} = 2$   $\text{Sp}(2N)$  gauge theory (with an antisymmetric hypermultiplet and four fundamental hypermultiplets), we have shown that representations corresponding to any pair of identical irreducible representations do. More generally, representations consisting of a direct sum of pairs of identical irreducible representations also correspond to vacua of this theory.

#### *Unbroken symmetry group of the vacuum solutions*

We now turn to the gauge enhancements for the solutions above, *i.e.*, the subgroup of the original  $\text{Sp}(2N)$  that remains unbroken by the scalar field vacuum expectation values  $\phi^i = m T^i$ . The analysis is similar to that of section 3.1.1.

Consider the solution consisting of two irreducible representations of dimension  $n$ . Then the elements  $U = e^H$  of  $\text{Sp}(2N)$  that obey  $U^\dagger T^i U = T^i$  are of the form  $H = h \otimes \mathbb{1}_n$ , where  $h^\dagger = -h$ . In the block-diagonal basis (3.6), the condition  $J H^T J = H$  becomes

$$g^* H^T g = H, \quad (4.12)$$

where  $g = i\sigma_y \otimes K'$  from (4.10). Equations (4.9) and (4.12) imply that  $(i\sigma_y) h^T (i\sigma_y) = h$ , that is,  $h \in \mathfrak{sp}(2)$ , giving  $\text{Sp}(2)$  gauge enhancement. Generalizing to  $2m$  identical blocks, and following the reasoning from section 3.1.1, one finds that the unbroken symmetry is  $\text{Sp}(2m)$ . This result holds irrespective of whether the blocks are even- or odd-dimensional. Hence, as noted above, some part of the gauge symmetry remains unbroken in all the classical vacua.<sup>5</sup>

#### 4.1.2 Case 2: $A = \tilde{A} = 0$

Now consider the classical vacua for which the vevs of the antisymmetric fields vanish,  $A = \tilde{A} = 0$ . Setting  $\phi^1 = \phi^2 = 0$  in (4.6), we find that  $\phi^3$ ,  $s$  and  $t$  are mutually commuting, and may be simultaneously diagonalized. In this diagonal basis, we have

$$\phi^3 = -\frac{M}{2} \begin{pmatrix} \mathbb{1}_r & & \\ & -\mathbb{1}_r & \\ & & 0 \end{pmatrix}, \quad s = Mm \begin{pmatrix} \mathbb{1}_r & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad t = Mm \begin{pmatrix} 0 & & \\ & -\mathbb{1}_r & \\ & & 0 \end{pmatrix}, \quad (4.13)$$

where, by (4.4), we have

$$J = \begin{pmatrix} 0 & \mathbb{1}_r & 0 \\ -\mathbb{1}_r & 0 & 0 \\ 0 & 0 & j \end{pmatrix}. \quad (4.14)$$

The rank  $r$  of the submatrices in (4.13) must be less than or equal to four, the number of fundamental hypermultiplets. This is because the F-term equations

$$\begin{aligned} \phi_a^b Q_b^I &= -\frac{1}{2} M Q_a^I \\ \tilde{Q}^{Ia} \phi_a^b &= -\frac{1}{2} M \tilde{Q}^{Ib}, \end{aligned} \quad (4.15)$$

have at most four independent eigenvectors, corresponding to  $I = 1$  to 4. The maximal rank  $r = 4$  is obtained, for example, by setting

$$\begin{aligned} Q_a^I &\sim \begin{cases} \delta_a^I \\ 0 \end{cases}, & \tilde{Q}^{Ia} &\sim \begin{cases} \delta^{Ia} \\ 0 \end{cases} & \text{for } \begin{cases} a = 1 \text{ to } 4 \\ a = 5 \text{ to } 8 \end{cases} \\ Q^{Ia} &= J^{ab} Q_b^I, & \tilde{Q}_a^I &= J_{ab} \tilde{Q}^{Ib}. \end{aligned} \quad (4.16)$$

#### 4.1.3 Case 3: General case

We now consider the possibility that the scalar components of both antisymmetric and fundamental hypermultiplet fields have non-vanishing vacuum expectation values. Although we have not completely classified the solutions to the F-term equations in this case, it seems likely that, for generic values of  $M$ , all the solutions decouple into direct sums of the solutions found in sections 4.1.1 and 4.1.2. Since the blocks with  $Q$  and  $\tilde{Q} \neq 0$  were of rank eight at most, they play a negligible role in the space of classical vacuum solutions of the  $\text{Sp}(2N)$  gauge theory in the large  $N$  limit. Since the large  $N$  limit is a necessary condition for the validity of the dual supergravity theory, we may neglect the fundamental hypermultiplets in this context, and focus on the vacuum solutions of sec. 4.1.1.

This neglect of the fundamental hypermultiplets is consistent with the computation of the beta function for the  $\text{Sp}(2N)$  gauge coupling. The four fundamental hypermultiplets make a contribution to the beta function that is of order  $1/N$  relative to that of the vector superfield and the antisymmetric hypermultiplet. Thus, we may consistently neglect vacua with non-zero vacuum expectation values of the scalar components of  $Q$  and  $\tilde{Q}$  in the large  $N$  limit.

Similar arguments apply to the  $\text{SU}(N)$  gauge theory with two antisymmetric and four fundamental hypermultiplets to be discussed in section 5.1.2.

## 4.2 Brane interpretation of vacua

The  $\mathcal{N} = 2$   $\text{Sp}(2N)$  gauge theory with one antisymmetric and four fundamental hypermultiplets is realized as the low-energy effective theory on a stack of D3-branes probing a type IIB background with orientifold group generated by  $\Omega' = \Omega(-1)^{F_L} R_{45}$ . This corresponds to the presence of an orientifold seven-plane lying along the 01236789 directions. For consistency one needs to add a stack of four D7-branes (and their mirrors) parallel to the orientifold plane.

The bosonic fields of this model (excluding those belonging to fundamental hypermultiplets) may be obtained as projections of the gauge fields  $A^\mu$  and adjoint scalars  $\phi^i$  of the  $\mathcal{N} = 4$   $\text{SU}(2N)$  model as follows. Consider a stack of  $2N$  D3-branes in an orientifold background, where the orientifold group acts via the  $2N \times 2N$  matrix  $\gamma_{\Omega'}$ . The orientifold action projects the fields according to

$$A^\mu = -\gamma_{\Omega'}(A^\mu)^T \gamma_{\Omega'}^{-1}, \quad \phi^{1,2} = +\gamma_{\Omega'}(\phi^{1,2})^T \gamma_{\Omega'}^{-1}, \quad \phi^3 = -\gamma_{\Omega'}(\phi^3)^T \gamma_{\Omega'}^{-1}. \quad (4.17)$$

For this theory,  $\gamma_{\Omega'} = J$  [20], so the gauge group is projected to  $\text{Sp}(2N)$ , and the projections (4.17) are identical to the field theory constraints (4.4).

This identification allows us to interpret the vacuum solutions in section 4.1.1 with D-brane configurations. The lack of any vacuum solution corresponding to a single irreducible representation (or to a direct sum of distinct irreducible representations) corresponds in the brane interpretation to the fact that the  $\mathbb{Z}_2$  projection of a two-sphere (or a set of spheres of different radii) by  $\Omega'$  does not give a closed two-surface, on which a D5-brane could be wrapped.

On the other hand, the solutions (4.11) consisting of a pair of identical (even- or odd-dimensional) irreducible  $\text{su}(2)$  representations correspond to a pair of spheres of equal radius on the covering space; by virtue of (4.10), these blocks are mapped to one another by the orientifold symmetry, so the pair of spheres are actually mirror images. Thus, the vacuum solutions (4.11) correspond to a single physical D5-brane with topology  $\mathbb{R}^4 \times S^2$ , *i.e.*, one D5-brane and its mirror. More generally, a solution containing  $2m$  identical even- or odd-dimensional representations corresponds to  $m$  D5-branes (whose associated spheres have equal radii) and their mirrors. As we saw above, the gauge enhancement for such a solution is  $\text{Sp}(2m)$ .

## 5 $\mathcal{N} = 2$ orbifold models with simple gauge groups

In this section, we consider mass deformations of  $\mathcal{N} = 2$  superconformal gauge theories that arise from IIB string theory in orientifolded orbifold backgrounds. Restricting our focus to models with a simple gauge group, there are only two such models [20] (see also [21]). These are  $\text{SU}(N)$  with one antisymmetric and one symmetric hypermultiplet, and  $\text{SU}(N)$  with two antisymmetric and four fundamental hypermultiplets. Both of these are orientifolds of  $\mathbb{Z}_2$  orbifolds, and both are examples of models “without vector structure” [22, 20].

### 5.1 Field theory vacuum states

### 5.1.1 $\mathcal{N} = 2$ orbifold model: $SU(N) + \square + \square$

The  $\mathcal{N} = 2$   $SU(N)$  gauge theory with hypermultiplets in the antisymmetric and symmetric representations arises as the field theory on a stack of  $2N$  D3-branes in a type IIB background with orientifold group  $(1 + \theta)(1 + \alpha\Omega')$  [20]. Here  $\theta = R_{6789}$  is the  $\mathbb{Z}_2$  orbifold generator, which acts as  $z_1 \rightarrow -z_1$  and  $z_2 \rightarrow -z_2$ . Also,  $\Omega' = \Omega(-1)^{F_L} R_{45}$ , and  $\alpha$  acts as  $z_1 \rightarrow iz_1$  and  $z_2 \rightarrow -iz_2$  (so  $\alpha^2 = \theta$ ).

As in sections 3 and 4, the bosonic fields of the gauge theory may be obtained as projections of the gauge fields  $A^\mu$  and adjoint scalars  $\Phi^i$  of an  $\mathcal{N} = 4$   $SU(2N)$  model. Consider a stack of  $2N$  D3 branes in an orientifold background, where the elements  $\theta$  and  $\alpha\Omega'$  of orientifold group<sup>8</sup> act on the D3-branes via the  $2N \times 2N$  matrices [20]

$$\gamma_\theta = i \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}, \quad \gamma_{\alpha\Omega'} = e^{i\pi/4} \begin{pmatrix} 0 & \mathbb{1}_N \\ i\mathbb{1}_N & 0 \end{pmatrix}. \quad (5.1)$$

The orbifold and orientifold projections impose the restrictions

$$A^\mu = \gamma_\theta A^\mu \gamma_\theta^{-1}, \quad A^\mu = -\gamma_{\alpha\Omega'} (A^\mu)^T \gamma_{\alpha\Omega'}^{-1}, \quad (5.2)$$

on the  $SU(2N)$  gauge field, projecting it to

$$A^\mu = \begin{pmatrix} a^\mu & 0 \\ 0 & (-a^\mu)^T \end{pmatrix}, \quad (5.3)$$

where  $a^\mu \in U(N) \sim SU(N) \times U(1)$ . (We ignore the  $U(1)$  factor, which in any case is suppressed for large  $N$ .) The scalar fields corresponding to D-brane positions are restricted to

$$\begin{aligned} \Phi^1 &= -\gamma_\theta \Phi^1 \gamma_\theta^{-1}, & \Phi^1 &= +i\gamma_{\alpha\Omega'} (\Phi^1)^T \gamma_{\alpha\Omega'}^{-1}, \\ \Phi^2 &= -\gamma_\theta \Phi^2 \gamma_\theta^{-1}, & \Phi^2 &= -i\gamma_{\alpha\Omega'} (\Phi^2)^T \gamma_{\alpha\Omega'}^{-1}, \\ \Phi^3 &= +\gamma_\theta \Phi^3 \gamma_\theta^{-1}, & \Phi^3 &= -\gamma_{\alpha\Omega'} (\Phi^3)^T \gamma_{\alpha\Omega'}^{-1}. \end{aligned} \quad (5.4)$$

The projected  $\Phi^i$ 's become

$$\Phi^1 = \begin{pmatrix} 0 & S \\ -\tilde{A} & 0 \end{pmatrix}, \quad \Phi^2 = \begin{pmatrix} 0 & A \\ \tilde{S} & 0 \end{pmatrix}, \quad \Phi^3 = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^T \end{pmatrix}. \quad (5.5)$$

Here  $S$  ( $A$ ) transforms in the symmetric  $\square\square$  (antisymmetric  $\square$ ) representation of  $SU(N)$ ;  $\tilde{S}$  and  $\tilde{A}$  transform in the corresponding conjugate representations.

Using (5.5), the superpotential

$$\frac{2\sqrt{2}}{g_{\text{YM}}^2} \text{tr}(\Phi^1 [\Phi^2, \Phi^3]), \quad (5.6)$$

---

<sup>8</sup>The orientifold group is actually generated by  $\alpha\Omega'$  alone, since  $(\alpha\Omega')^2 = \theta$  (which in terms of the  $\gamma$ 's translates into  $\gamma_\theta = -\gamma_{\alpha\Omega'} (\gamma_{\alpha\Omega'}^T)^{-1}$ ), but it is useful to see the action of the orbifold generator separately.



inherited from the  $SU(2N)$   $\mathcal{N} = 4$  model, is projected to

$$\mathcal{W} = \frac{4\sqrt{2}}{g_{\text{YM}}^2} \text{tr}(\tilde{S}\phi S + \tilde{A}\phi A). \quad (5.7)$$

The same projection when applied to the usual  $SU(2N)$  mass deformation (2.2), however, gives vanishing masses for the  $S$  and  $A$  fields. Instead, we use the alternative mass deformation

$$\mathcal{W}_{\text{mass}} = \frac{2\sqrt{2}}{g_{\text{YM}}^2} \text{tr}[m_\phi(\Phi^3)^2 + m\Phi^1\Phi^2], \quad (5.8)$$

which projects to

$$\mathcal{W}_{\text{mass}} = \frac{2\sqrt{2}}{g_{\text{YM}}^2} \text{tr}(2m_\phi\phi^2 + m\tilde{S}S - m\tilde{A}A). \quad (5.9)$$

The projection of the deformation (5.8) to the  $SU(N) + \square + \square\square$  theory thus leads to the restriction  $m_S = -m_A = m$ , although in general the masses need not be related in this way. The sum of the superpotentials (5.7) and (5.9) then gives rise to the F-term equations

$$\begin{aligned} A\tilde{A} + S\tilde{S} &= -2m_\phi\phi, \\ \phi S + S\phi^T &= -mS, \\ \phi A + A\phi^T &= +mA, \\ \tilde{S}\phi + \phi^T\tilde{S} &= -m\tilde{S}, \\ \tilde{A}\phi + \phi^T\tilde{A} &= +m\tilde{A}. \end{aligned} \quad (5.10)$$

### *Vacuum solutions*

We now turn to finding solutions to the vacuum equations (5.10). First we solve the F-term equations of the fields  $\Phi^i$  before imposing the projections (5.4). If we redefine

$$\Phi^1 = -i(\phi^1 - i\phi^2), \quad \Phi^2 = -i(\phi^1 + i\phi^2), \quad \Phi^3 = -i\phi^3, \quad (5.11)$$

and set  $m_\phi = m$ , then the superpotential (5.6) and (5.8) leads to the F-term equations

$$[\phi^i, \phi^j] = -m\epsilon^{ijk}\phi^k. \quad (5.12)$$

The general solution is, as before,  $\phi^i = mT^i$ . Hence we may write

$$\Phi^1 = -imT^- = -im(T^x - iT^y), \quad \Phi^2 = -imT^+ = -im(T^x + iT^y), \quad \Phi^3 = -imT^z, \quad (5.13)$$

where the  $T^i$  is any (in general reducible) even-dimensional representation of the  $\mathfrak{su}(2)$  algebra. Equation (5.13) also satisfies the D-term equations  $\sum_{i=1}^3 [\Phi^i, (\Phi^i)^\dagger] = 0$ .

Next we determine *which*  $\mathfrak{su}(2)$  representations (5.13) survive the projections (5.4), *i.e.*, can be cast into the form (5.5). Consider first the  $2n$ -dimensional irreducible representation  $T_{2n}^i$ . In the standard basis, (A.2) and (A.3) do not have the form (5.5). However, if we relabel the rows and columns as follows:  $n+1 \rightarrow 1$ ,  $n+2 \rightarrow n+2$ ,  $n+3 \rightarrow 3$ ,  $n+4 \rightarrow n+4$ ,

up through  $2n$ , and  $n \rightarrow n+1$ ,  $n-1 \rightarrow 2$ ,  $n-2 \rightarrow n+3$ ,  $n-3 \rightarrow 4$ , down through 1, then the  $\Phi^i$ 's in (5.13) are exactly of the form (5.5) with

$$\begin{aligned}
\phi &= m \begin{pmatrix} -\frac{1}{2} & & & & & \\ & \frac{3}{2} & & & & \\ & & -\frac{5}{2} & & & \\ & & & \frac{7}{2} & & \\ & & & & \ddots & \\ & & & & & (-)^n(n - \frac{1}{2}) \end{pmatrix}, \\
S &= 2m \begin{pmatrix} a_0 & & & & & \\ & 0 & a_2 & & & \\ & a_{-2} & 0 & & & \\ & & & 0 & a_4 & \\ & & & a_{-4} & 0 & \\ & & & & & \ddots \end{pmatrix}, \\
A &= 2m \begin{pmatrix} 0 & a_{-1} & & & & \\ a_1 & 0 & & & & \\ & & 0 & a_{-3} & & \\ & & a_3 & 0 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix}, \tag{5.14}
\end{aligned}$$

where the  $\mathfrak{su}(2)$  commutation relations require  $a_k^2 = \frac{1}{4}(n^2 - k^2)$ . The signs of the  $a_k$  must be chosen to satisfy  $a_{-k} = (-)^k a_k$  in order that  $S$  and  $A$  be symmetric and antisymmetric respectively. This further implies that  $\tilde{S} = S$  and  $\tilde{A} = A$ , which is consistent with  $\Phi^2 = (\Phi^1)^\dagger$ . (Similarly, a representation consisting of a *direct sum* of such even-dimensional irreducible representations can be written in the form (5.5).)

Although a single irreducible odd-dimensional representation of  $\mathfrak{su}(2)$  cannot be of the form (5.5), consider a solution (5.13) in which  $T^i$  contains a pair of  $d$ -dimensional ( $d = 2n+1$ ) irreducible representations in block-diagonal form

$$T^i = \left( \begin{array}{c|c} T_{2n+1}^i & 0 \\ \hline 0 & (-T_{2n+1}^i)^T \end{array} \right), \tag{5.15}$$

where  $T_{2n+1}^i$  are in the standard form (A.2), (A.3). Specifically,

$$\Phi^3 = m \begin{pmatrix} n & & & & & \\ & n-1 & & & & \\ & & \ddots & & & \\ & & & -n & & \\ \hline & & & & -n & \\ & & & & & -n+1 \\ & & & & & & \ddots \\ & & & & & & & n \end{pmatrix},$$

$$\begin{aligned}
\Phi^1 &= 2m \left( \begin{array}{cccc|cccc} 0 & & & & & & & \\ a_{n-\frac{1}{2}} & 0 & & & & & & \\ & a_{n-\frac{3}{2}} & 0 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & & & & & \end{array} \right), \\
\Phi^2 &= (\Phi^1)^\dagger,
\end{aligned} \tag{5.16}$$

where  $a_k^2 = b_k^2 = \frac{1}{4}(n-k+\frac{1}{2})(n+k+\frac{1}{2})$  but the choice of signs is undetermined. Clearly, (5.16) is not of the form (5.5) but can be made so by relabeling the rows and columns as follows:  $1 \rightarrow 1$ ,  $2 \rightarrow 2d-1$ ,  $3 \rightarrow 3$ ,  $4 \rightarrow 2d-3$ , up through  $d \rightarrow d$ , and  $d+1 \rightarrow d+1$ ,  $d+2 \rightarrow d-1$ ,  $d+3 \rightarrow d+3$ ,  $d+4 \rightarrow d-3$ , up through  $2d \rightarrow 2d$ . Then  $\Phi^i$  will be exactly of the form (5.5) with

$$\begin{aligned}
\phi &= m \begin{pmatrix} n & & & & & \\ & n-1 & & & & \\ & & n-2 & & & \\ & & & \ddots & & \\ & & & & -n+1 & \\ & & & & & -n \end{pmatrix}, \\
S &= 2m \begin{pmatrix} & & & & 0 & \\ & & & & 0 & -b_{\frac{1}{2}-n} \\ & & & 0 & a_{n-\frac{3}{2}} & \\ & & \ddots & \ddots & & \\ & 0 & -b_{n-\frac{3}{2}} & & & \\ 0 & a_{\frac{1}{2}-n} & & & & \end{pmatrix}, \\
A &= 2m \begin{pmatrix} & & & & a_{n-\frac{1}{2}} & 0 \\ & & & -b_{\frac{3}{2}-n} & 0 & \\ & & \ddots & \ddots & & \\ & a_{n-\frac{3}{2}} & 0 & & & \\ -b_{n-\frac{1}{2}} & 0 & & & & \\ 0 & & & & & \end{pmatrix}.
\end{aligned} \tag{5.17}$$

The choice of signs  $b_{n-\frac{1}{2}-k} = (-)^k a_{n-\frac{1}{2}-k}$  then makes  $S$  symmetric and  $A$  antisymmetric, and also yields  $\tilde{S} = S$  and  $\tilde{A} = A$ .

To conclude, we have exhibited two classes of classical vacua: those corresponding to any number of even-dimensional  $\mathfrak{su}(2)$  representations, and those corresponding to pairs of identical odd-dimensional  $\mathfrak{su}(2)$  representations.

### Unbroken symmetry group of the vacuum solutions

Let us consider the gauge enhancement for each of the solutions  $\Phi^i$  described above. As before, we seek traceless, antihermitian matrices  $H$  that commute with  $\Phi^i$ , and that satisfy the constraints (5.2) on the adjoint representation of the gauge group, namely

$$H = \gamma_\theta H \gamma_\theta^{-1}, \quad H = -\gamma_{\alpha\Omega'} H^T \gamma_{\alpha\Omega'}^{-1}. \quad (5.18)$$

*i.e.*,  $H$  must take the form (5.3).

For a single even-dimensional representation, Schur's lemma implies that  $H \propto \mathbb{1}$ , and (5.18) then implies that  $H = 0$ ; the gauge symmetry is completely broken. The same result holds for a direct sum of *distinct* (even-dimensional) representations.

Consider  $m$  identical  $2n$ -dimensional representations,  $T_{2n}^i \otimes \mathbb{1}_m$ , where  $T_{2n}^i$  is related to  $\Phi^i$  by (5.13) and  $\Phi^i$  is of the form (5.5) with (5.14). Schur's lemma shows that  $H = \mathbb{1}_{2n} \otimes h$ , where  $h \in \mathfrak{u}(m)$ . Eq. (5.18) then implies that  $h \in \mathfrak{so}(m)$ , and the gauge enhancement is  $\mathrm{SO}(m)$ .

Next consider a pair of identical  $(2n+1)$ -dimensional representations,  $\mathbb{1}_2 \otimes T_{2n+1}^i$ , where  $T_{2n+1}^i$  has the standard form (A.2), (A.3). Schur's lemma implies  $H = h \otimes \mathbb{1}_{2n+1}$ , where  $h \in \mathfrak{u}(2)$ . The unitary transformation

$$U = \begin{pmatrix} \mathbb{1}_{2n+1} & 0 \\ 0 & K^T L \end{pmatrix}, \quad (5.19)$$

where  $L = \mathrm{diag}(1, 1, -1, -1, 1, 1, -1, -1, \dots)$  puts the reducible representation into the form (5.15) (the presence of  $L$  generates the relative signs  $b_{n-\frac{1}{2}-k} = (-)^k a_{n-\frac{1}{2}-k}$  between the two irreducible representations), and puts  $H$  into the form

$$H = \begin{pmatrix} h_{11} \mathbb{1}_{2n+1} & h_{12} K^T L \\ h_{21} L K & h_{22} \mathbb{1}_{2n+1} \end{pmatrix}. \quad (5.20)$$

Finally, rearranging rows and columns as described just below eq. (5.16) and then imposing eq. (5.18) yields  $h_{12} = h_{21} = 0$  and  $h_{22} = -h_{11}$ , so that  $h \in \mathfrak{u}(1)$ . Generalizing this procedure to  $2m$  representations of dimension  $(2n+1)$ , we find that  $h \in \mathfrak{u}(m)$ , so the unbroken gauge symmetry is  $\mathrm{U}(m)$ .

In summary, for  $2m$  odd-dimensional (resp.  $m$  even-dimensional) blocks of the same dimension, the gauge enhancement is  $\mathrm{U}(m)$  (resp.  $\mathrm{SO}(m)$ ).<sup>5</sup>

#### 5.1.2 $\mathcal{N} = 2$ orbifold model: $\mathrm{SU}(N) + 2 \square + 4 \square$

The  $\mathcal{N} = 2$   $\mathrm{SU}(N)$  gauge theory with two antisymmetric hypermultiplets and four fundamental hypermultiplets arises as the field theory on a stack of D3-branes in a type IIB background with orientifold group  $(1 + \theta)(1 + \Omega')$  and four D7-branes for consistency [20]. The analysis of this model is very similar to that in section 5.1.1. The bosonic fields of the gauge theory (excluding those corresponding to fundamental hypermultiplets) may be obtained as projections of the gauge fields  $A^\mu$  and adjoint scalars  $\Phi^i$  of the  $\mathcal{N} = 4$   $\mathrm{SU}(2N)$

model. Consider a stack of  $2N$  D3 branes in an orientifold background, where the independent generators  $\theta$  and  $\Omega'$  of the orientifold group act on the D3-branes via the  $2N \times 2N$  matrices [20]

$$\gamma_\theta = i \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}, \quad \gamma_{\Omega'} = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}. \quad (5.21)$$

The orbifold and orientifold projections impose the restrictions

$$A^\mu = \gamma_\theta A^\mu \gamma_\theta^{-1}, \quad A^\mu = -\gamma_{\Omega'} (A^\mu)^T \gamma_{\Omega'}^{-1}, \quad (5.22)$$

on the  $SU(2N)$  gauge field, thus projecting it to

$$A^\mu = \begin{pmatrix} a^\mu & 0 \\ 0 & (-a^\mu)^T \end{pmatrix}, \quad (5.23)$$

where  $a^\mu \in U(N) \sim SU(N) \times U(1)$ . (We ignore the  $U(1)$  factor, which in any case is suppressed for large  $N$ .) The scalar fields corresponding to D-brane positions are restricted to

$$\begin{aligned} \Phi^1 &= -\gamma_\theta \Phi^1 \gamma_\theta^{-1}, & \Phi^1 &= +\gamma_{\Omega'} (\Phi^1)^T \gamma_{\Omega'}^{-1}, \\ \Phi^2 &= -\gamma_\theta \Phi^2 \gamma_\theta^{-1}, & \Phi^2 &= +\gamma_{\Omega'} (\Phi^2)^T \gamma_{\Omega'}^{-1}, \\ \Phi^3 &= +\gamma_\theta \Phi^3 \gamma_\theta^{-1}, & \Phi^3 &= -\gamma_{\Omega'} (\Phi^3)^T \gamma_{\Omega'}^{-1}. \end{aligned} \quad (5.24)$$

The projected  $\Phi^i$ 's become

$$\Phi^1 = \begin{pmatrix} 0 & A_2 \\ -\tilde{A}_1 & 0 \end{pmatrix}, \quad \Phi^2 = \begin{pmatrix} 0 & A_1 \\ \tilde{A}_2 & 0 \end{pmatrix}, \quad \Phi^3 = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^T \end{pmatrix}. \quad (5.25)$$

Here the  $A_i$ 's transform in the antisymmetric  $\square$  representation of  $SU(N)$  and the  $\tilde{A}_i$ 's in the conjugate representation.

The projected fields obtained in this way do not include the hypermultiplet fields in the fundamental representation, whose scalar components could in principle have non-zero vacuum expectation values. However, we expect that vacua with non-zero fundamental vevs will not be relevant in the large  $N$  limit, for reasons discussed in section 4.1.3.

The superpotential for this model is exactly the same as in section 5.1.1, namely, (5.6), (5.8), with the addition of some terms for the fundamental hypermultiplets. The form of the mass deformation (5.8) yields  $m_2 = -m_1 = m$  for the masses of the two antisymmetric hypermultiplets, although this need not hold in general. If we set the vacuum expectation values of the fundamental superfields to zero, the F-term equations for  $SU(N) + 2\square + 4\square$  are given by (5.10), replacing  $A$  with  $A_1$  and  $S$  with  $A_2$ .

#### *Vacuum solutions*

The analysis of the solutions to the F-term equations (with the fundamental superfield vevs vanishing) is similar to that of section 5.1.1. Unlike the  $\mathcal{N} = 2$   $SU(N) + \square + \square\square$  theory, a single even-dimensional irreducible representation cannot be put into the form (5.25). Essentially, this is because it is not possible to distribute an odd number of  $a_n$ 's in (A.3)

into two antisymmetric matrices. Similarly, no solutions consisting of *distinct* irreducible representations of  $\mathfrak{su}(2)$  is possible. Thus, as in the  $\mathrm{Sp}(N) + \square + 4\square$  theory, there are no classically massive vacua.

Solutions (5.15)-(5.17) built from pairs of odd-dimensional irreducible representations *can* be put into the form (5.25) if we set  $b_{n-\frac{1}{2}-k} = a_{n-\frac{1}{2}-k}$ , and replace  $A$  with  $A_1$  and  $S$  with  $A_2$ . This choice of signs also implies  $\tilde{A}_1 = A_1$  and  $\tilde{A}_2 = -A_2$ .

Solutions can also be constructed from a pair of identical even-dimensional irreducible representations, as follows. Consider the  $2n$ -dimensional representation of the  $\mathrm{SU}(N) + \square + \square$  theory, namely, eq. (5.5) with (5.14), but replacing  $A$  with  $S_1$  and  $S$  with  $S_2$ . Choosing  $a_{-n} = a_n$  (which differs from section 5.1.1) makes both  $S_1$  and  $S_2$  symmetric, and implies  $\tilde{S}_1 = -S_1$  and  $\tilde{S}_2 = S_2$ , since  $\Phi^2 = (\Phi^1)^\dagger$ . Take the tensor product of this representation with  $\mathbb{1}_2$ , and then perform the unitary transformation  $U = \mathrm{diag}(\mathbb{1}_{2n}, \mathbb{1}_n \otimes i\sigma_y)$  to obtain

$$\Phi^1 = \begin{pmatrix} 0 & S_2 \otimes i\sigma_y \\ \tilde{S}_1 \otimes i\sigma_y & 0 \end{pmatrix}, \Phi^2 = \begin{pmatrix} 0 & S_1 \otimes i\sigma_y \\ -\tilde{S}_2 \otimes i\sigma_y & 0 \end{pmatrix}, \Phi^3 = \begin{pmatrix} \phi \otimes \mathbb{1}_2 & 0 \\ 0 & -\phi^T \otimes \mathbb{1}_2 \end{pmatrix}. \quad (5.26)$$

These matrices have the correct form (5.25) for a vacuum solution for  $\mathrm{SU}(N)$  with two antisymmetric hypermultiplets, with  $A_i = S_i \otimes i\sigma_y$ . This result generalizes to a solution with any even number of identical even-dimensional representations.

To conclude, the vacua of the  $\mathrm{SU}(N) + \square + \square$  theory correspond to (a direct sum of) pairs of identical irreducible representations of  $\mathfrak{su}(2)$ ; the pairs can consist of either even- or odd-dimensional representations.

We observe that the relation between the solutions of the  $\mathrm{SU}(N) + 2\square + 4\square$  theory and those of the  $\mathrm{SU}(N) + \square + \square$  theory is analogous to that between the  $\mathrm{Sp}(2N) + \square + 4\square$  theory and the  $\mathrm{Sp}(2N) + \square$  (=adjoint) theory (*i.e.*, the pure  $\mathcal{N} = 4$   $\mathrm{Sp}(2N)$  theory).

#### *Unbroken symmetry group of the vacuum solutions*

Let us consider the gauge enhancement for each the solutions  $\Phi^i$  described above. Again, we seek traceless, antihermitian matrices  $H$  that commute with  $\Phi^i$ , and that satisfy the constraints (5.22) on the adjoint representation of the gauge group, namely

$$H = \gamma_\theta H \gamma_\theta^{-1}, \quad H = -\gamma_{\Omega'} H^T \gamma_{\Omega'}^{-1} \quad (5.27)$$

*i.e.*,  $H$  must take the form (5.23).

Consider the vacuum solution consisting of  $2m$  identical  $2n$ -dimensional representations of  $\mathfrak{su}(2)$ ,  $T_{2n}^i \otimes \mathbb{1}_{2m}$ , where  $T_{2n}^i$  is related to  $\Phi^i$  by (5.13) and  $\Phi^i$  is of the form (5.5) with (5.14). Schur's lemma implies that  $H = \mathbb{1}_2 \otimes \mathbb{1}_n \otimes h$ , where  $h \in \mathfrak{u}(2m)$ . The unitary transformation  $U = \mathrm{diag}(\mathbb{1}_{2mn}, \mathbb{1}_n \otimes J)$ , where  $J = i\sigma_y \otimes \mathbb{1}_m$ , puts  $H$  into the form

$$H = \begin{pmatrix} \mathbb{1}_n \otimes h & 0 \\ 0 & \mathbb{1}_n \otimes (-JhJ) \end{pmatrix}. \quad (5.28)$$

Eq. (5.27) then implies that  $JhJ = h^T$ , that is,  $h \in \mathfrak{sp}(2m)$ , so that the unbroken symmetry group is  $Sp(2m)$ .

The analysis of the unbroken gauge symmetry of the solution consisting of  $2m$  identical  $(2n+1)$ -dimensional representations is completely analogous to subsection 5.1.1, except that the unitary transformation (5.19) does not include the matrix  $L$  (since the relative sign between the irreducible representations is now  $b_{n-\frac{1}{2}-k} = a_{n-\frac{1}{2}-k}$ ). The conclusion is unaltered, and the unbroken gauge symmetry is  $U(m)$ .

In summary, for  $2m$  odd-dimensional (resp.  $2m$  even-dimensional) blocks of the same dimension, the gauge enhancement is  $U(m)$  (resp.  $Sp(2m)$ ).<sup>5</sup>

## 5.2 Brane interpretation of the vacua

We now turn to the brane interpretations of the vacuum solutions found in subsections 5.1.1 and 5.1.2. The coordinates on the cover space of the orientifolds are the  $\Phi^i$ 's (5.5), (5.25) and are the most convenient coordinates to use when discussing the interpretation of the vacuum solutions.

Before proceeding, however, let us also mention that a general method for dealing with branes at singularities was pioneered in ref. [23]. This method is inherently field-theoretic and can be used to derive the nature of the singularity from the field theory alone. Performing such an analysis for the model in section 5.1.1 (in the absence of the mass deformation) leads to the invariant coordinates (in the 6789 directions)  $x = -S\tilde{A}$ ,  $y = A\tilde{S}$ . In the process one also introduces the auxiliary coordinate  $w = -A\tilde{A}$ . The F-term equations imply that these coordinates satisfy the equation  $xy = w^2$ , which describes the  $\mathbb{Z}_2$  orbifold singularity<sup>9</sup>. The relation between  $x, y$  and the  $\Phi^i$ 's is as follows. For the 6789 directions  $\Phi^{1,2}$  transforms under the orbifold transformation as  $\Phi^{1,2} \rightarrow -\Phi^{1,2}$ ; thus, the invariant coordinates may be taken to be  $(\Phi^1)^2$  and  $(\Phi^2)^2$ . These are block-diagonal matrices

$$(\Phi^1)^2 = \begin{pmatrix} -S\tilde{A} & 0 \\ 0 & -(-S\tilde{A})^T \end{pmatrix}, \quad (\Phi^2)^2 = \begin{pmatrix} A\tilde{S} & 0 \\ 0 & -(A\tilde{S})^T \end{pmatrix}, \quad (5.29)$$

in which the upper block corresponds to the physical branes and the lower block corresponds to their orientifold mirrors. The invariant coordinates corresponding to the 6789 directions can thus be taken to be  $x = -S\tilde{A}$  and  $y = A\tilde{S}$ , the same expressions as above. In addition, for the 45 directions,  $\phi$  may be interpreted as the invariant coordinate; the lower block in  $\Phi^3$ ,  $-\phi^T$ , represents the coordinate of the mirror under the orientifold transformation and the orbifold generator does not act on the 45 directions. The virtue of the invariant coordinates is that the presence of the singularity is explicit.

Let us now return to the interpretation of vacuum states in terms of D-brane configurations. The fields  $\Phi^i$  that solve the F- and D-term equations can be transformed into block-diagonal form (2.6), where each block satisfies the Casimir relation

$$\frac{1}{2}(\Phi^1\Phi^2 + \Phi^2\Phi^1) + (\Phi^3)^2 = -m^2 \left[ (T^x)^2 + (T^y)^2 + (T^z)^2 \right] = -m^2 c_2(n) \mathbb{1}, \quad (5.30)$$

where  $n$  is the dimension of the block. The hermitian matrices  $X^i = -iT^i$  can be interpreted (for each block) as the coordinates for a non-commutative two-sphere of radius  $\sqrt{-c_2(n)}$  ( $\sim \frac{1}{2}n$  for large  $n$ ) on the covering space.

<sup>9</sup>To see this note that one may parameterize the equation as  $x = a^2$ ,  $y = b^2$  and  $w = ab$ . This parameterization has a redundancy since it is unchanged under the  $\mathbb{Z}_2$  transformation  $(a, b) \rightarrow -(a, b)$ .

The solution to the  $SU(N) + \square + \square\square$  gauge theory consisting of  $m$  identical irreducible even-dimensional  $\mathfrak{su}(2)$  representations (5.14) corresponds to  $m$  copies of  $S^2$  on the covering space. The generators of the orientifold group act *within* each block and consequently on the corresponding sphere, yielding  $m$  copies of  $S^2/\mathbb{Z}_4$ . The brane picture is therefore analogous to that for  $m$  even-dimensional (odd-dimensional) representations in the  $\mathcal{N} = 4$   $Sp(2N)$  ( $SO(N)$ ) gauge theories, with the consequent  $SO(m)$  gauge enhancement.

The solution to the  $SU(N) + 2\square + 4\square$  gauge theory consisting of  $2m$  identical irreducible *even-dimensional*  $\mathfrak{su}(2)$  representations (direct sum of eq. (5.26)) corresponds to  $2m$  copies of  $S^2$  on the covering space. Consider this as  $m$  pairs of spheres. The orbifold generator  $\theta$  (when transformed into the basis in which the solution is block-diagonal) acts within each block, but the generator  $\Omega'$  (similarly transformed) exchanges the blocks within each pair. Consequently, under the orientifold group projection, we are left with  $m$  copies of  $S^2/\mathbb{Z}_2$ . The brane picture here is therefore analogous to that for  $2m$  odd-dimensional (even-dimensional) representations in the  $\mathcal{N} = 4$   $Sp(2N)$  ( $SO(N)$ ) gauge theories, with the consequent  $Sp(2m)$  gauge enhancement.

The solutions to the  $SU(N) + 2\square + 4\square$  gauge theory consisting of  $2m$  identical irreducible *odd-dimensional*  $\mathfrak{su}(2)$  representations (5.15)-(5.17) also corresponds to  $2m$  copies of  $S^2$  on the covering space. Again, consider this as  $m$  pairs of spheres. In this case, however, *all* of the (non-trivial) elements of the orientifold group (when transformed into the basis in which the solution is block-diagonal) act non-trivially on each pair (specifically, they act on the two-dimensional space as  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$ ) as well as acting within each block. The brane interpretation in this case is not completely transparent, but observe that the orientifold group acts on the set of  $2m$  D5-branes in the same way that the matrices  $\gamma_\theta$ ,  $\gamma_{\Omega'}$  and  $\gamma_\theta\gamma_{\Omega'}$  defined in eq. (5.21) act on  $2N$  D3-branes, leading to an analogous gauge enhancement: in the latter case,  $U(N)$ , and in the former,  $U(m)$ .

A similar story holds for the solutions to the  $SU(N) + \square + \square\square$  gauge theory consisting of  $2m$  identical irreducible odd-dimensional  $\mathfrak{su}(2)$  representations.

## 6 Conclusions

In this paper we have determined the classical vacuum states of  $\mathcal{N} = 1^*$  mass deformations of  $\mathcal{N} = 4$  and  $\mathcal{N} = 2$  conformal models with simple gauge groups. The undeformed theories arise as the effective field theories on D3 branes in various orientifold and/or orbifold backgrounds. The classical vacua were found from solutions of the F- and D-term equations of the mass-deformed superpotentials, the gauge enhancements of these vacua were analyzed, and interpretations in terms of D5-branes were suggested. These results are collected in Table 1, which lists the building blocks of the vacuum solutions of each of the gauge theories studied, together with the resulting gauge enhancements and brane interpretation. (The last column specifies the topology of the D5-branes, suppressing the  $\mathbb{R}^4$  part.)



Gauge theory	Vacuum solution	Enhancement	Interpretation
$\mathrm{Sp}(2N) + \text{adjoint}$	$2m$ odd-dimensional irreps	$\mathrm{Sp}(2m)$	$m$ $S^2$ 's
	$m$ even-dimensional irreps	$\mathrm{SO}(m)$	$m$ $\mathbb{RP}^2$ 's
$\mathrm{SO}(N) + \text{adjoint}$	$2m$ even-dimensional irreps	$\mathrm{Sp}(2m)$	$m$ $S^2$ 's
	$m$ odd-dimensional irreps	$\mathrm{SO}(m)$	$m$ $\mathbb{RP}^2$ 's
$\mathrm{Sp}(2N) + \square + 4\square$	$2m$ irreps	$\mathrm{Sp}(2m)$	$m$ $S^2$ 's
$\mathrm{SU}(N) + \square + \square\square$	$2m$ odd-dimensional irreps	$\mathrm{U}(m)$	see text
	$m$ even-dimensional irreps	$\mathrm{SO}(m)$	$m$ $S^2/\mathbb{Z}_4$ 's
$\mathrm{SU}(N) + 2\square + 4\square$	$2m$ odd-dimensional irreps	$\mathrm{U}(m)$	see text
	$2m$ even-dimensional irreps	$\mathrm{Sp}(2m)$	$m$ $S^2/\mathbb{Z}_2$ 's

**Table 1:** Gauge enhancements and D-brane interpretations of  $\mathcal{N} = 1^*$  vacua.

The solutions consisting of  $2m$  identical odd-dimensional (resp. even-dimensional) representations of the  $\mathcal{N} = 4$   $\mathrm{Sp}(2N)$  (resp.  $\mathrm{SO}(N)$ ) gauge theories are new, as well as the classification of vacua for various mass-deformed  $\mathcal{N} = 2$  models with simple gauge groups.

The F-equations for all cases (with vanishing hypermultiplet fields in the fundamental representation) are equal to the F-equations of the  $\mathcal{N} = 4$  theory with  $\mathrm{SU}(N)$  gauge group, supplemented by additional conditions specific to the particular model being considered. One therefore expects the first-order supergravity equations to be given by projections of those of Polchinski and Strassler [10]. In the large  $N$  limit, we argue that hypermultiplets in the fundamental representations essentially decouple and therefore may be neglected (see sections 4 and 5.1.2 for the relevant models). Consequently, in the large  $N$  limit, all the  $\mathcal{N} = 1^*$  theories treated here allow a brane interpretation of the classical vacua in terms of D5 branes wrapped on closed two-surfaces. Since the large  $N$  limit is a necessary condition for the supergravity dual, and hypermultiplets in the fundamental representation decouple in this limit, the models studied here do not have a baryonic Higgs phase in the dual supergravity description.

## Acknowledgments

We would like to thank Steve Gubser, Igor Klebanov, Joe Polchinski, Matt Strassler, and Andy Waldron for conversations.

## Appendices

### A Some facts about $\mathfrak{su}(2)$ representations

One choice for the commutation relations of the real Lie algebra  $\mathfrak{su}(2)$  is

$$[T^i, T^j] = -\epsilon^{ijk} T^k. \quad (\text{A.1})$$

The unitary representations of  $\mathfrak{su}(2)$  are given in terms of  $n \times n$  anti-hermitian matrices, and there is one irreducible representation for each integer  $n \geq 1$ . A standard basis for the generators of these representations is

$$T_n^z = i \begin{pmatrix} j & & & & \\ & j-1 & & & \\ & & \ddots & & \\ & & & -j+1 & \\ & & & & -j \end{pmatrix}, \quad (\text{A.2})$$

where  $j$  is an integer or a half-integer, with  $n = 2j + 1$ . Both  $T_n^x$  and  $T_n^y$  have non-zero entries only directly above and directly below the diagonal with  $T_n^x$  symmetric (and hence purely imaginary) and  $T_n^y$  anti-symmetric (and hence real):

$$T_n^x = i \begin{pmatrix} 0 & a_{j-\frac{1}{2}} & & & \\ a_{j-\frac{1}{2}} & 0 & a_{j-\frac{3}{2}} & & \\ & a_{j-\frac{3}{2}} & 0 & \ddots & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad T_n^y = \begin{pmatrix} 0 & a_{j-\frac{1}{2}} & & & \\ -a_{j-\frac{1}{2}} & 0 & a_{j-\frac{3}{2}} & & \\ & -a_{j-\frac{3}{2}} & 0 & \ddots & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \quad (\text{A.3})$$

The commutator  $[T^x, T^y] = -T^z$  implies that  $2(-a_{j-m+\frac{1}{2}}^2 + a_{j-m-\frac{1}{2}}^2) = j - m$ , with  $a_{j+\frac{1}{2}} = a_{-j-\frac{1}{2}} = 0$ , from which one obtains  $a_k^2 = \frac{1}{4}(j-k+\frac{1}{2})(j+k+\frac{1}{2})$ . The choice of the signs of  $a_k$ , however, remains arbitrary. The standard quantum mechanics convention (for the hermitian generators  $J^i = -iT^i$ ) is to take  $a_k > 0$ , but we will occasionally need to make other choices (see section 5.1.1).

### B Some technical details

Here we explain how some of the assertions made in the paper about vacuum solutions of the gauge theories are proved. For the  $\mathcal{N} = 4$   $\text{Sp}(2N)$  ( $\text{SO}(N)$ ) gauge theory, it was claimed in section 3 that, for a solution consisting of a direct sum of *distinct* even-dimensional (odd-dimensional) irreducible  $\mathfrak{su}(2)$  representations, the matrix  $g$  ( $f$ ) is block-diagonal, with each block proportional to  $K$  (3.5). In the following, we will concentrate on the  $\text{Sp}(2N)$  case; the analysis for  $\text{SO}(N)$  is similar. First, we apply the condition (3.4) to  $T^i$ , where each  $T_{n_k}^i$  block takes the standard form (A.2), (A.3). The equation  $(T^z)^T g = -g T^z$  implies that, within each diagonal block of  $g$ , only entries along the reverse diagonal can be non-zero. The  $T^z$  condition would permit certain non-zero entries in the blocks of  $g$  not along the diagonal,

but these entries vanish when we impose  $(T^x)^T g = -g T^x$  (assuming that the off-diagonal blocks connect representations of different dimensions); thus,  $g$  is block diagonal. Finally,  $(T^x)^T g = -g T^x$  implies that the diagonal blocks of  $g$  are proportional to

$$K = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \ddots & & & \end{pmatrix}, \quad (\text{B.1})$$

provided that the signs of the  $a_k$  in (A.3) are chosen to satisfy  $a_k = a_{-k}$  (the choice adopted in sections 3 and 4). The condition arising from  $T^y$  imposes no further restrictions. The condition  $gg^* = -\mathbb{1}$  then implies that each block equals  $K$  up to a phase, which may be removed by a unitary transformation.

If the direct sum contains a pair of identical  $\text{su}(2)$  representations, then the relevant part of  $q$  has the form

$$\begin{pmatrix} g_{11} & g_{12} \\ -g_{12}^T & g_{22} \end{pmatrix}, \quad (\text{B.2})$$

where the equation  $(T^z)^T g = -gT^z$  implies that only entries along the reverse diagonal of each matrix  $g_{ij}$  can be non-zero. The condition  $(T^x)^T g = -gT^x$  then implies that each  $g_{ij}$  is proportional to  $K$ .

When we have  $m$  even-dimensional representations of the same dimension, we find, repeating the above analysis pairwise for the constituent blocks and using the fact that  $K^T = -K$  when even-dimensional, that the relevant part of  $g$  is equal (after removing some phases) to a real symmetric matrix tensored with  $K$ . The real symmetric matrix (tensored by  $K$ ) may be diagonalized by an orthogonal matrix (tensored by  $\mathbb{1}$ ) that does not affect the  $T^i$ 's into the form  $\text{diag}(K, \dots, K)$  (using the condition  $gg^* = -\mathbb{1}$ ). When we have  $m$  odd-dimensional representations the analysis is similar except that, since  $K$  is symmetric when odd-dimensional, one finds that the relevant part of  $g$  is given by an antisymmetric matrix tensored with  $K$ . The condition  $gg^* = -\mathbb{1}$  then implies that  $m$  must be even (since an odd-dimensional real antisymmetric matrix cannot square to  $-\mathbb{1}$ ), and one may transform  $g$  into block-diagonal form, with each block proportional to (3.8) with  $K^T = K$ .

For the  $\mathcal{N} = 2$   $\text{Sp}(2N) + \square + 4\square$  gauge theory, the conditions (3.4) are replaced by (4.8). For a solution consisting of a direct sum of a pair of identical  $\text{su}(2)$  representations, where  $g$  has the form

$$\begin{pmatrix} g_{11} & g_{12} \\ -g_{12}^T & g_{22} \end{pmatrix}, \quad (\text{B.3})$$

the equation  $(T^z)^T g = -g T^z$  implies, as before, that only entries along the reverse diagonal of each matrix  $g_{ij}$  can be non-zero. The modified condition  $(T^x)^T g = g T^x$  then implies that each  $g_{ij}$  is proportional, not to  $K$ , but to  $K'$  (4.9). Since  $g_{ii}$  must be antisymmetric, while  $K'$  is symmetric, the diagonal blocks of  $g$  must vanish, and we are left with  $g$  of the form (4.10). The generalization to the case when one has  $2m$  representations of equal dimensions is straightforward.

## References

- [1] J. Maldacena, “The large  $N$  limit of superconformal field theories and supergravity.” *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200](#); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory.” *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109](#); E. Witten, “Anti-de Sitter space and holography.” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150](#).
- [2] S. Kachru and E. Silverstein, “4d conformal theories and strings on orbifolds.” *Phys. Rev. Lett.* **80** (1998) 4855–4858, [hep-th/9802183](#); A. Lawrence, N. Nekrasov, and C. Vafa, “On conformal field theories in four dimensions.” *Nucl. Phys.* **B533** (1998) 199–209, [hep-th/9803015](#); Y. Oz and J. Terning, “Orbifolds of  $AdS_5 \times S_5$  and 4d conformal field theories.” *Nucl. Phys.* **B532** (1998) 163, [hep-th/9803167](#); S. Gukov, “Comments on  $\mathcal{N} = 2$  AdS orbifolds.” *Phys. Lett.* **B439** (1998) 23–28, [hep-th/9806180](#).
- [3] Z. Kakushadze, “Gauge theories from orientifolds and large  $N$  limit.” *Nucl. Phys.* **B529** (1998) 157–179, [hep-th/9803214](#); Z. Kakushadze, “On large  $N$  gauge theories from orientifolds.” *Phys. Rev.* **D58** (1998) 106003, [hep-th/9804184](#).
- [4] A. Fayyazuddin and M. Spalinski, “Large  $N$  superconformal gauge theories and supergravity orientifolds.” *Nucl. Phys.* **B535** (1998) 219–232, [hep-th/9805096](#).
- [5] E. Witten, “Baryons and branes in anti de Sitter space.” *JHEP* **07** (1998) 006, [hep-th/9805112](#).
- [6] O. Aharony, A. Fayyazuddin, and J. Maldacena, “The large  $N$  limit of  $\mathcal{N} = 2, 1$  field theories from three-branes in F-theory.” *JHEP* **07** (1998) 013, [hep-th/9806159](#).
- [7] S. Gukov and A. Kapustin, “New  $\mathcal{N} = 2$  superconformal field theories from M/F theory orbifolds.” *Nucl. Phys.* **B545** (1999) 283–308, [hep-th/9808175](#).
- [8] I. P. Ennes, C. Lozano, S. G. Naculich, and H. J. Schnitzer, “Elliptic models, type IIB orientifolds and the AdS/CFT correspondence.” *Nucl. Phys.* **B591** (2000) 195–226, [hep-th/0006140](#).
- [9] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, “The supergravity dual of  $\mathcal{N} = 1$  super Yang-Mills theory.” *Nucl. Phys.* **B569** (2000) 451–469, [hep-th/9909047](#).
- [10] J. Polchinski and M. J. Strassler, “The string dual of a confining four-dimensional gauge theory.” [hep-th/0003136](#).
- [11] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity.” *Nucl. Phys.* **B536** (1998) 199, [hep-th/9807080](#); I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and  $(\chi)$ SB-resolution of naked singularities.” *JHEP* **08** (2000) 052, [hep-th/0007191](#); K. Pilch and N. P. Warner, “ $\mathcal{N} = 1$  supersymmetric renormalization

- group flows from IIB supergravity,” [hep-th/0006066](#); J. M. Maldacena and C. Nunez, “Towards the large  $N$  limit of pure  $\mathcal{N} = 1$  super Yang Mills.” [hep-th/0008001](#); C. Vafa, “Superstrings and topological strings at large  $N$ .” [hep-th/0008142](#); M. Graña and J. Polchinski, “Supersymmetric three-form flux perturbations on  $\text{AdS}_5$ .” *Phys. Rev. D* **63** (2001) 026001, [hep-th/0009211](#).
- [12] O. Aharony and A. Rajaraman, “String theory duals for mass-deformed  $\text{SO}(N)$  and  $\text{USp}(2N)$   $\mathcal{N} = 4$  SYM theories.” *Phys. Rev.* **D62** (2000) 106002, [hep-th/0004151](#).
  - [13] C. Vafa and E. Witten, “A strong coupling test of S duality.” *Nucl. Phys.* **B431** (1994) 3–77, [hep-th/9408074](#); R. Donagi and E. Witten, “Supersymmetric Yang-Mills theory and integrable systems.” *Nucl. Phys.* **B460** (1996) 299–334, [hep-th/9510101](#).
  - [14] R. C. Myers, “Dielectric-branes.” *JHEP* **12** (1999) 022, [hep-th/9910053](#).
  - [15] V. G. Kac and A. V. Smilga, “Normalized vacuum states in  $\mathcal{N} = 4$  supersymmetric Yang-Mills quantum mechanics with any gauge group.” *Nucl. Phys.* **B571** (2000) 515–554, [hep-th/9908096](#).
  - [16] D. Kabat and W. Taylor IV, “Linearized supergravity from matrix theory.” *Phys. Lett.* **B426** (1998) 297–305, [hep-th/9712185](#).
  - [17] A. Hanany, B. Kol, and A. Rajaraman, “Orientifold points in M theory.” *JHEP* **10** (1999) 027, [hep-th/9909028](#).
  - [18] E. G. Gimon and J. Polchinski, “Consistency conditions for orientifolds and D-manifolds.” *Phys. Rev.* **D54** (1996) 1667–1676, [hep-th/9601038](#).
  - [19] A. Sen, “F-theory and orientifolds.” *Nucl. Phys.* **B475** (1996) 562–578, [hep-th/9605150](#); T. Banks, M. R. Douglas, and N. Seiberg, “Probing F-theory with branes.” *Phys. Lett.* **B387** (1996) 278–281, [hep-th/9605199](#); O. Aharony, J. Sonnenschein, S. Yankielowicz, and S. Theisen, “Field theory questions for string theory answers.” *Nucl. Phys.* **B493** (1997) 177–197, [hep-th/9611222](#); M. R. Douglas, D. A. Lowe, and J. H. Schwarz, “Probing F-theory with multiple branes.” *Phys. Lett.* **B394** (1997) 297–301, [hep-th/9612062](#).
  - [20] J. Park and A. M. Uranga, “A note on superconformal  $\mathcal{N} = 2$  theories and orientifolds.” *Nucl. Phys.* **B542** (1999) 139–156, [hep-th/9808161](#).
  - [21] J. Erlich, A. Hanany, and A. Naqvi, “Marginal deformations from branes.” *JHEP* **03** (1999) 008, [hep-th/9902118](#).
  - [22] E. Witten, “Toroidal compactification without vector structure.” *JHEP* **02** (1998) 006, [hep-th/9712028](#).
  - [23] M. R. Douglas and G. Moore, “D-branes, Quivers, and ALE Instantons.” [hep-th/9603167](#); M. R. Douglas, B. R. Greene, and D. R. Morrison, “Orbifold resolution by D-branes.” *Nucl. Phys.* **B506** (1997) 84–106, [hep-th/9704151](#).